Change of numéraire

Math 622

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1 Introduction

1.1 Another look at the Black-Scholes risk neutral model

Let $r > 0$ be the constant risk free rate. So far, we've considered the following Black-Scholes model of a stock:

$$dS_t = \alpha S_t dt + \sigma S_t dW_t,$$

where α is a constant and W_t a Brownian motion.
To price any financial derivative based on S, the first question we have to answer is: what is the risk neutral probability measure? In other words, we want to find a probability measure $\widetilde{P}$ such that $e^{-rt}S_t$ is a martingale under $\widetilde{P}$.
We're used to looking at $e^{-rt}S_t$ as the discounted stock price. And the risk neutral measure is interpreted as the probability such that the discounted stock price is a martingale.
There is yet a slightly different way of looking at this. If we denote

$$\begin{aligned} dNt &= rNtdt \\ N_0 &= 1, \end{aligned}$$

that is $N_t = e^{rt}$; then N_t is the price of one unit of the *money market account.* Then $e^{-rt}S_t$ is nothing but the price of the stock *expressed in the unit of the money market account.* The risk neutral measure above can be looked at as the probability such that the price of the stock, *expressed in the unit of the money market account*, is a martingale.
Note that there is another asset, which is also a martingale (albeit a trivial one), when expressed in the unit of the money market account: the money market price

process itself. It is clear that the price of the money market is 1 when expressed under its own unit, thus it is a (trivial) martingale.

1.2 Main questions of this chapter

The process N_t in the above is a *numéraire*, and the risk neutral measure we've studied in the Black-Scholes model is the risk neutral measure associated to the (domestic) money market numéraire. To re-emphasize, it is the probability measure such that the price of all non-dividend paying assets are martingales when expressed in the unit of the domestic money market account.

It is clear that the domestic money market is not the only choice for a numéraire. In a world where there is a foreign currency, then the foreign money market is also a possible choice of numéraire. The obvious question is, *how do we determine the risk neutral probability associated with the foreign money market numéraire*? More generally, how do we decide a risk neutral probability associated with any numéraire, as long as we have a model for that particular numéraire? (for example, the zero coupon bond $B(t, T)$ can also be used as numéraire.) This chapter will address these questions.

One important thing to note about the choice of numéraire: we shall take only *non-dividend* paying assets as numéraire. Using this criterion, the domestic currency itself **cannot** be used as a numéraire. This point may not be obvious, but observe that the currency's dynamics is:

$$dC_t = 0dt = R_t C_t dt - R_t C_t dt,$$

where C denotes the price of a unit of domestic currency and R_t is the interest rate at time t. (This is just a complicated way to express that C_t is a constant).

On the other hand, the dynamics of a unit of the domestic money market M_t is

$$dM_t = R_t M_t dt.$$

Therefore, (see also Shreve's Sections 5.5.1 and 5.5.2) the currency can be looked at as the price of a share of the money market, paying dividend at rate R_t.

1.3 New set up of this chapter

Since in this chapter, we will introduce the foreign exchange rate and foreign money market, it is natural that we are into a multiple risky assets setting. Moreover, the risk free rate *will no longer be a constant* r. We will consider a risk-free rate process $R(t)$ that can be stochastic. The associated discount process is

$$D_t = \exp\Big\{ - \int_0^t R(u)du \Big\},$$

and for the domestic money market risk neutral measure $\widetilde{P}$, we will require that $D_t S_t$ be a martingale under $\widetilde{P}$. Also, as we use different numéraires, there will be different risk-neutral measures corresponding to these numéraires. It is important to clarify which risk-neutral probability we are discussing. For example, we will call the risk neutral measure associated with the domestic money market *the domestic risk neutral measure.* Similarly, we will call the risk neutral measure associated with the foreign money market *the foreign risk neutral measure.* In this note, by dollars we also mean the domestic currency and vice versa.
To prepare for these new set ups, we will review a few details on stochastic calculus and multi-asset model in the next couple sections.

1.4 Why study change of numéraire

(i) A risk neutral pricing formula when the financial product is quoted in foreign currency:
Suppose we have a Euro style derivative that pays V_T (in foreign currency) at time T. We want to find the no-arbitrage price V_0 of this derivative at time 0. Let $R^f(t)$ be the foreign interest rate, which is an adapted process. Intuitively, the pricing formula would be

$$V_0 = \widetilde{E}^f\left[e^{-\int_0^T R^f(u)du} V_T\right],$$

where $\widetilde{E}^f$ is a foreign risk neutral measure. How to define this $\widetilde{E}^f$ so that the above formula holds is a question that we will address in this chapter.

(ii) Modeling when the interest rate is random:

When the interest rate is random, the pricing formula for a Euro-style derivative on a stock S_t becomes complicated (See formula 9.4.6 in Shreve and the discussion after). However, when we use a suitable numéraire, which is the zero-coupon bond in this case, the pricing formula becomes much simpler (formula 9.4.7 in Shreve). Thus this suggests one should model S_t under the risk-neutral measure associated with the zero-coupond bond (called the T-forward measure). Indeed, it turns out that the correct object to model is the forward price of the stock S_t (Section 9.4.3 in Shreve - We will also discuss this in the next Lecture). The point is that our usual choice of numéraire (the domestic money market) may not be the best choice in all situations. Studying the change of numéraire suggests other choice of numéraire that would simplify the problem, both in terms of pricing and in terms of modeling.

2 Markets with multiple risky assets

Itô process models for markets with multiple risky assets are treated in Chapter 5 of Shreve. This is a brief review.

Consider a market with m risky assets. Prices are given in a domestic currency, which, for convenience, we will assume to be US dollars. A price model consists of a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, a filtration $\{\mathcal{F}(t);\, t \geq 0\}$, and an m-vector-valued stochastic process $S(t) = (S_1(t), \ldots, S_m(t))$ that represents the asset prices and that is adapted to the filtration. The goal of modeling is to construct S so that its statistical behavior approximately matches what is actually observed in the market. The return of asset i over the small interval of time $[t, t + dt]$ is given by $\dfrac{dS_i(t)}{S_i(t)}$.

From analysis of the historical data or from structural models for the market, the modeler can generate estimates for all assets of

(i) the mean rate, local of change of asset i: $\quad \mu_i(t)\, dt = E\left[\dfrac{dS_i(t)}{S_i(t)} \;\Big|\; \mathcal{F}(t)\right]$;

(ii) the local (square) volatility of asset i: $\sigma_i^2(t)\, dt = \text{Var}\left(\dfrac{dS_i(t)}{S_i(t)} \;\Big|\; \mathcal{F}(t)\right)$; and

(iii) the correlation between the returns of asset i and j, $i \neq j$;

$$\rho_{ij}(t), dt = \frac{\text{Cov}\left(\frac{dS_i(t)}{S_i(t)}, \frac{dS_j(t)}{S_j(t)} \;\Big|\; \mathcal{F}(t)\right)}{\sigma_i(t)\sigma_j(t)}.$$

A nice way to construct models that can fit these informally described parameters, and for which the price processes are continuous, is to use stochastic differential equations driven by a multi-dimensional Brownian motion $W(t) = (W_1(t), \ldots, W_d(t))$. Suppose we set

$$\frac{dS_i(t)}{S_i(t)} = \mu_i(t)\, dt + \sum_{k=1}^{d} \sigma_{ik}(t)\, dW_k(t), \qquad 1 \leq i \leq m. \tag{1}$$

Let $\{\mathcal{F}(t);\, t \geq 0\}$ be a filtration for W and assume $\mu_i(t)$ and $\sigma_{ij}(t)$, $t \geq 0$ are adapted to $\{\mathcal{F}(t);\, t \geq 0\}$. Recall that, by definition, $W_1, \ldots, W_d$ are independent

Brownian motions. Then, formally, $E[dW_i(t) \mid \mathcal{F}(t)] = 0$, $E[(dW_i(t))^2 \mid \mathcal{F}(t)] = dt$ and $E[dW_i(t))\, dW_j(t) \mid \mathcal{F}(t)] = 0$. Thus, for each asset,

$$E\left[\frac{dS_i(t)}{S_i(t)} \;\Big|\; \mathcal{F}(t)\right] = \mu_i(t)\,dt + \sum_{k=1}^{d} \sigma_{ik}(t) E[dW_k(t) \mid \mathcal{F}(t)] = \mu_i(t)\,dt,$$

in conformity with (i). On the other hand,

$$\begin{aligned} \mathrm{Var}\left(\frac{dS_i(t)}{S_i(t)} \;\Big|\; \mathcal{F}(t)\right) &= E\left[\left(\sum_{k=1}^{d} \sigma_{ik}(t)\,dW_k(t)\right)^2 \;\Big|\; \mathcal{F}(t)\right] \\ &= \left[\sum_{k=1}^{d} \sigma_{ik}^2(t)\right] dt. \end{aligned} \tag{2}$$

By a similar calculation

$$\begin{aligned} \mathrm{Cov}\left(\frac{dS_i(t)}{S_i(t)}, \frac{dS_j(t)}{S_j(t)} \;\Big|\; \mathcal{F}(t)\right) &= E\left[\sum_{k=1}^{d} \sigma_{ik}(t)\,dW_k(t) \cdot \sum_{l=1}^{d} \sigma_{jl}(t)\,dW_l(t) \;\Big|\; \mathcal{F}(t)\right] \\ &= \left[\sum_{k=1}^{d} \sigma_{ik}(t)\sigma_{jk}(t)\right] dt. \end{aligned} \tag{3}$$

Therefore, we can match the model (1) to the variances and correlations prescribed in (ii) and (iii) by choosing d and $\sigma_{ij}(t)$, $1 \le i, j \le m$, so that

$$\begin{aligned} \sum_{k=1}^{d} \sigma_{ik}^2(t) &= \sigma_i^2(t) \\ \sum_{k=1}^{d} \sigma_{ik}(t)\sigma_{jk}(t) &= \rho_{ij}(t)\sigma_i(t)\sigma_j(t) \end{aligned}$$

Because model (1) is flexible enough to capture price means, volatilities, and correlations in this manner, it is a standard model for multi-asset markets. Usually, it is expressed in the more familiar form

$$dS_i(t) = \mu_i(t)S_i(t)\,dt + S_i(t)\sum_{k=1}^{d} \sigma_{ik}(t)\,dW_k(t). \tag{4}$$

Example 1. Given $\sigma_1(t)$, $\sigma_2(t)$, and $\rho(t)$ satisfying $-1 \le \rho(t) \le 1$, we want to construct a model with two risky assets so that the volatility process of S_1 is $\sigma_1(t)$, that of S_2 is $\sigma_2(t)$ and

$$\rho(t) = \frac{\mathrm{Cov}\left(\frac{dS_1(t)}{S_1(t)}, \frac{dS_2(t)}{S_2(t)}\right)}{\sigma_1(t)\sigma_2(t)}.$$

This can be achieved with the model

$$dS_1(t) = \mu_1(t)S_1(t)\,dt + \sigma_1(t)S_1(t)\,dW_1(t), \tag{5}$$

$$dS_2(t) = \mu_2(t)S_2(t)\,dt + \sigma_2(t)S_2(t)\left[\rho(t)\,dW_1(t) + \sqrt{1-\rho^2(t)}\,dW_2(t)\right]. \tag{6}$$

Indeed, equation (5) is by itself just the usual model for an asset with volatility process $\sigma_1(t)$. As for S_2,

$$\begin{aligned}
\mathrm{Var}\left(\frac{dS_2(t)}{S_2(t)}\;\Big|\;\mathcal{F}(t)\right) &= E\left[\left(\sigma_2(t)\left[\rho(t)\,dW_1(t) + \sqrt{1-\rho^2(t)}\,dW_2(t)\right]\right)^2\;\Big|\;\mathcal{F}(t)\right]\\
&= \sigma_2^2(t)\left[\rho^2(t)E[(dW_1(t))^2] + (1-\rho^2(t))E[(dW_2(t))^2] + 2\rho(t)\sqrt{1-\rho^2(t)}E[dW_1(t)\,dW_2(t)]\right]\\
&= \sigma_2^2(t)\left[\rho^2(t)\,dt + (1-\rho^2(t))\,dt\right] = \sigma_2^2(t)\,dt
\end{aligned}$$

Also,

$$\begin{aligned}
\mathrm{Cov}\left(\frac{dS_1(t)}{S_1(t)}, \frac{dS_2(t)}{S_2(t)}\;\Big|\;\mathcal{F}(t)\right) &= E\left[\sigma_1(t)\,dW_1(t)\cdot\left(\sigma_2(t)\rho(t)\,dW_1(t) + \sqrt{1-\rho^2(t)}\,dW_2(t)\right)\;\Big|\;\mathcal{F}(t)\right]\\
&= \sigma_1(t)\sigma_2(t)\left[\rho(t)E[(dW_1(t))^2] + \sqrt{1-\rho^2(t)}E[dW_1(t)dW_2(t)]\right]\\
&= \sigma_1(t)\sigma_2(t)\rho(t)\,dt
\end{aligned}$$

2.1 Market with foreign currency

Example 1 can be translated into a model for a market with one risky asset and a tradable foreign currency, which is the important setting we want to discuss in this chapter.
Let $S(t)$ be the price in dollars of the risky asset.
Let $Q(t)$ denote the price (in dollars) of one unit of the foreign currency at time t. Thus $Q(t)$ is the exchange rate. Q can be thought of as a second risky asset; it fluctuates randomly and these fluctuations are generally correlated with those of S. Hence the model of Example 1 is appropriate.
Following the notation in Shreve, (9.3.1)-(9.3.2), it shall be written:

$$dS(t)\,dt = \alpha(t)S(t)\,dt + \sigma_1(t)S(t)\,dW_1(t), \tag{7}$$

$$dQ(t)\,dt = \gamma(t)Q(t)\,dt + \sigma_2(t)Q(t)\left[\rho(t)\,dW_1(t) + \sqrt{1-\rho^2(t)}\,dW_2(t)\right]. \tag{8}$$

Suppose that in this market one can purchase a foreign money market account earning the risk-free rate with one's foreign cash.
Let $R^f(t)$, $t \geq 0$, be the risk-free foreign rate. We will say that one unit of this money market is an account in which one unit of foreign currency is deposited at time $t = 0$ and never withdrawn.
Thus the price at time t in *foreign currency* of one foreign money market unit is

$$M^f(t) = \exp\{\int_0^t R^f(u)\,du\}.$$

The price at time t of one foreign money market unit *in dollars* is

$$N^f(t) = M^f(t)Q(t).$$

If we are investing in this market, we will certainly deposit any idle *foreign cash* in the *foreign money market*; otherwise, we forego the interest we could earn at rate R^f. Thus it is really more appropriate to write the model in terms of $S(t)$ and $N^f(t)$. An easy calculation shows that this model is:

$$\begin{aligned} dS(t)\,dt &= \alpha(t)S(t)\,dt + \sigma_1(t)S(t)\,dW_1(t), && (9) \\ dN^f(t)\,dt &= \left[\gamma(t) + R^f(t)\right] N^f(t)\,dt \\ &\quad + \sigma_2(t)N^f(t)\left[\rho(t)\,dW_1(t) + \sqrt{1-\rho^2(t)}\,dW_2(t)\right]. && (10) \end{aligned}$$

Notes:
(i) Both equations (9), (10) are expressed in terms of dollars, not the foreign currency; and N^f is, again, the price of the foreign money market in dollars.

(ii) Both equations (9), (10) are not written in risk neutral measure setting.

3 A review of multi-dimensional stochastic calculus

This is more review material, collected for convenience of reference.

3.1 Multi-dimensional stochastic integration

Let $W(t) = (W_1(t), \dots, W_d(t))$ be a d-dimensional Brownian motion, and let $\{\mathcal{F}(t);\, t \geq 0\}$ be a filtration for W. Let the vector-valued process $\Theta(t) = (\theta_1(t), \dots, \theta(t))$ be adapted to $\{\mathcal{F}(t);\, t \geq 0\}$. For d-dimensional vectors, we use $x \cdot y = \sum_{k=1}^{d} x_i y_i$ to denote the inner product and $\|x\| = \sqrt{\sum_{k=1}^{d} x_k^2} = \sqrt{x \cdot x}$ to denote the norm of a vector. Accordingly, we use the following convenient notation:

$$\int_0^t \Theta(u) \cdot\, dW(u) = \sum_{k=1}^{d} \int_0^t \theta_k(u)\, dW_k(u).$$

For example, if we define $\underline{\sigma_i}(t) = (\sigma_{i1}(t), \dots, \sigma_{id}(t))$, the equation for $S_i(t)$ in (4) can be written

$$dS_i(t) = \mu_i(t) S_i(t)\, dt + S_i(t) \left[\underline{\sigma_i}(t) \cdot dW(t) \right].$$

3.2 Linear stochastic differential equations

The following general fact is useful. The solution to the stochastic differential equation $dX(t) = \mu(t) X(t)\, dt + X(t)[\Theta(t) \cdot dW(t)]$ is

$$X(t) = X(0) \exp\{ \int_0^t \Theta(u) \cdot dW(u) - \frac{1}{2} \int_0^t \|\Theta(u)\|^2\, du + \int_0^t \mu(u)\, du \}. \tag{11}$$

To show this expression is a solution requires just an application of the multi-dimensional Itô rule. We will not show it is the unique solution; this is done in the theory of stochastic differential equations.
A simple calculation also shows that X solves

$$dX(t) = \mu(t) X(t)\, dt + X(t)[\Theta(t) \cdot dW(t)] \tag{12}$$

if and only if

$$d\left[e^{-\int_0^t \mu(u)\, du} X(t) \right] = \left[e^{-\int_0^t \mu(u)\, du} X(t) \right] [\Theta(t) \cdot dW(t)] \tag{13}$$

We will pass between these two equivalent equations frequently and without comment.

3.3 Girsanov's theorem

Let

$$Z(t)\exp\{-\int_0^t \Theta(u)\cdot dW(u) - \frac{1}{2}\int_0^t \|\Theta(u)\|^2\,du\}.$$

If it is assumed that $E[Z(T)] = 1$, then

$$\mathbf{P}^Z(A) = E[\mathbf{1}_A Z(T)], \quad A \in \mathcal{F},$$

defines a new probability measure. The multi-dimensional Girsanov theorem says that

$$W^Z(t) = W(t) + \int_0^t \Theta(u)\,du = \left(W_1(t) + \int_0^t \theta_1(u)\,du, \ldots, W_d(t) + \int_0^t \theta_d(u)\,du\right)$$

is a Brownian motion up to time T under $\mathbf{P}^Z$.

3.4 What happens when we use a general $\mathcal{F}_t^W$ martingale Z_t in the change of measure fomula

Suppose that $Z(t)$ is a $\mathcal{F}(t)$ martingale and $Z(0) = 1$. It follows that $E[Z(T)] = Z(0) = 1$. We can still define a new measure

$$\mathbf{P}^Z(A) = E[\mathbf{1}_A Z(T)], \quad A \in \mathcal{F},$$

as above (the measure $\mathbf{P}^Z$ is well-defined). However, this is a bit abstract. We did not impose any dynamics on Z_t. But we still want to learn, for example, the distribution of $W(t)$ under $\mathbf{P}^Z$. It turns out that when the filtration is generated by the Brownian motion, then the martingale representation will give us information about the dynamics of Z_t and the Girsanov's theorem will tell us about the behavior of W_t under P^Z.

(i) Martingale representation:
Assume now that the filtration $\{\mathcal{F}(t);\, t \geq 0\}$ is generated by W. Under this **important** assumption, if $Z(t)$ is a martingale with respect to $\{\mathcal{F}(t);\, t \geq 0\}$ under measure $\mathbf{P}$, then the martingale representation theorem says there exists an adapted, vector valued process $\Gamma(t) = (\gamma_1(t), \ldots, \gamma_d(t))$ such that

$$Z(t) = Z(0) + \int_0^t \Gamma(u)\cdot dW(t).$$

Suppose that $Z(0) = 1$ and that $Z(t) > 0$ for all $0 \le t \le T$ almost surely. By defining $\nu(t) = (\nu_1(t), \ldots, \nu_d(t)) = \frac{1}{Z(t)}\Gamma(t)$, one can write

$$Z(t) = 1 + \int_0^t Z(u)\frac{1}{Z(u)}\Gamma(u) \cdot dW(u) = 1 + \int_0^t Z(u)[\nu(u) \cdot dW(u)].$$

It then follows from equation (11) that

$$Z(t) = \exp\{\int_0^t \nu(u) \cdot dW(u) - \frac{1}{2}\int_0^t \|\nu(u)\|^2 \, du\}. \tag{14}$$

(ii) Girsanov's theorem:
By applying Girsanov's theorem to this expression we obtain:

Theorem 1. *Suppose that* $\{\mathcal{F}(t);\, t \ge 0\}$ *is generated by* W *and that* $Z(t)$ *is an* $\{\mathcal{F}(t);\, t \ge 0\}$*-martingale under* $\mathbf{P}$ *such that* $E[Z(t)] = 1$ *for all* t*. Define* $P^Z(A) = E[\mathbf{1}_A Z(T)]$, $A \in \mathcal{F}$.
Suppose in addition that $Z(T) > 0$ *almost surely. Then there is an* $\{\mathcal{F}(t);\, t \ge 0\}$*-adapted process* $\nu(t) = (\nu_1(t), \ldots, \nu_d(t))$ *so that equation (14) holds, and for this process,*

$$W^Z(t) = W(t) - \int_0^t \nu(u)\, du \quad \textit{is a Brownian motion up to time } T \textit{ under } P^Z.$$

The only point we have not justified (and will not) is that if $Z(T) > 0$ almost surely, then $Z(t) > 0$ for all $t \le T$ almost surely.
This theorem is essentially Theorem 9.2.1 in Shreve; we have just stated it more generally. It is *one of the important theorems* in this Chapter. Later on, we will replace Z_t with $D_t N_t$, where D_t is the discounted process mentioned above and N_t the actual numéraire we want to study (for example, the domestic or foreign money market). Then P^Z is the risk neutral measure associated with that numéraire. And this theorem tells us how the distribution of the Brownian motion changes under this risk neutral measure. Note that at this level, the Theorem is a bit abstract: it only tells us that the process ν exists—it does not say how to find ν. In applications, one can often determine ν from other assumptions, as we shall see in studying numéraires.

4 The domestic risk-neutral measure

Consider the model for $S(t) = (S_1(t), \ldots, S_m(t))$ given in equation (4). Assume henceforth that $\{\mathcal{F}(t);\, t \geq 0\}$ is the filtration generated by W. This allows us to employ the martingale representation theorem.
Add also to the model a risk-free rate process $R(t)$, $t \geq 0$, which is assumed to be non-negative and adapted to $\{\mathcal{F}(t);\, t \geq 0\}$. The associated discount process is denoted by $D(t) = \exp\{-\int_0^t R(u)\, du\}$.
The price of S_i, in terms of the domesitc money market, is $D_t S_i(t)$. We have the following important definition:

Definition 4.1. *The domestic risk-neutral measure for the model (4) is the probability measure $\widetilde{P}$ such that $D_t S_i(t)$ is a martingale under $\widetilde{P}$, for all i.*

The model (4) is equivalent to

$$d\,[D(t)S_i(t)] = (\mu_i(t) - R(t))D(t)S_i(t)\, dt + D(t)S_i(t) \sum_{k=1}^{d} \sigma_{ik}(t)\, dW_k(t), \qquad 1 \leq i \leq m, \tag{15}$$

as an easy calculation shows; (compare to equations (12) and (13)). We are interested in finding a domestic risk-neutral measure, assuming one exists. The essential ingredient is provided in the following theorem, which reviews material from Chapter 5 of Shreve. This review is useful because the procedure of finding the risk-neutral measure is a template for changing measure for numéraires.

Theorem 2. *Assume that there is a risk-neutral measure $\widetilde{\mathbf{P}}$ for model (15) given by $\widetilde{\mathbf{P}}(A) = E[\mathbf{1}_A Z]$, where Z is an $\mathcal{F}(T)$ measurable random variable for which $E[Z] = 1$ and $\mathbf{P}(Z > 0) = 1$. Then*

$$Z = \exp\left\{-\int_0^T \Theta(u) \cdot dW(u) - \frac{1}{2}\int_0^T \|\Theta(u)\|^2\, du\right\}, \tag{16}$$

where $\Theta(t) = (\theta_1(t), \ldots, \theta_d(t))$ is an $\{\mathcal{F}(t);\, t \geq 0\}$-adapted process that is a solution of the market price of risk equation

$$\begin{pmatrix} \sigma_{11}(t) & \sigma_{12}(t) & \cdots & \sigma_{1d}(t) \\ \sigma_{21}(t) & \sigma_{22}(t) & \cdots & \sigma_{2d}(t) \\ \vdots & \vdots & & \vdots \\ \sigma_{m1}(t) & \sigma_{m2}(t) & \cdots & \sigma_{md}(t) \end{pmatrix} \cdot \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \\ \vdots \\ \theta_d(t) \end{pmatrix} = \begin{pmatrix} \mu_1(t) - R(t) \\ \mu_2(t) - R(t) \\ \vdots \\ \mu_m(t) - R(t) \end{pmatrix}, \qquad 0 \leq t \leq T. \tag{17}$$

If this equation has a unique solution, the risk-neutral measure is unique. Under $\widetilde{\mathbf{P}}$,

$$\widetilde{W}(t) = \left(W_1(t) + \int_0^t \theta_1(u)\, du, \ldots, W_d(t) + \int_0^t \theta_d(u)\, du \right) \tag{18}$$

is a Brownian motion up to time t.

Proof: The process $Z(t) = E[Z \mid \mathcal{F}(t)]$ is a martingale and since Z is $\mathcal{F}(T)$-measurable, $Z(T) = Z$. By Theorem 1, there is an adapted process ν such that $Z(t) = \exp\{\int_0^t \nu(u) \cdot dW(u) - \frac{1}{2} \int_0^t \|\nu(u)\|^2\, du\}$. Equation (16) then follows if we set $\Theta(t) = -\nu(t)$. By Girsanov, the process $\widetilde{W}$ defined in equation (18) is a Brownian motion up to time T under $\widetilde{\mathbf{P}}$. From (18), $dW_i(t) = d\widetilde{W}_i(t) - \theta_i(t)\, dt$. By using this substitution in the equation (15) for $D(t)s_i(t)$,

$$d\left[D(t)S_i(t)\right] = \left(\mu_i(t) - R(t) - \sum_{k=1}^{d} \sigma_{ik}(t)\theta_k(t) \right) D(t)S_i(t)\, dt + D(t)S_i(t) \sum_{k=1}^{d} \sigma_{ik}(t)\, d\widetilde{W}_k(t). \tag{19}$$

This must be a martingale under the risk-neutral measure $\widetilde{\mathbf{P}}$ for all i; that is what it means for $\widetilde{\mathbf{P}}$ to be a risk-neutral measure. Thus the 'dt' term in (19) must be 0 for each i:

$$\sum_{k=1}^{d} \sigma_{ik}(t)\theta_k(t) = \mu_i(t) - R(t), \qquad 1 \le i \le m.$$

The matrix form of these equations is just equation (17) of the theorem statement. This completes the proof. ◇

As a consequence of the proof, the stochastic differential equation model for the discounted prices under the risk-neutral measure is

$$d\left[D(t)S_i(t)\right] = D(t)S_i(t) \sum_{k=1}^{d} \sigma_{ik}(t)\, d\widetilde{W}_k(t), \qquad 1 \le i \le m.$$

Remarks:

1) The equations summarized by (17) are called the *market price of risk equations.* The difference $\mu_i(t) - R(t)$ can be regarded as a risk premium; it is the amount by which the expected rate of gain of the asset is larger than the risk-free rate. Investors typically demand $\mu_i(t) - R(t)$ to be positive before investing in i, to make up for the fact that the investment carries risk. The expression $\mu_i(t) - R(t) = \sum_{k=1}^{d} \sigma_{ik}(t)\theta_k(t)$ may be thought of as a decomposition of

$\mu_i(t) - R(t)$ into a sum contributions from each source of random fluctations of $S_i(t)$; $\theta_k(t)$ is effectively a price per unit of volatility of the contribution $\sigma_{ik}(t)\theta_k(t)$.

2. Theorem 2 implies that a necessary condition for the existence of a risk-neutral measure is that (17) must have a solution $\Theta(t)$. However, having a solution to (17) is not by itself a sufficient condition for the existence of a risk neutral measure. If $\Theta(t)$ is a solution and $Z = \exp\left\{-\int_0^T \Theta(u)\cdot dW(u) - \frac{1}{2}\int_0^T \|\Theta(u)\|^2\,du\right\}$, one must check in addition that $E[Z] = 1$, in order that $\widetilde{\mathbf{P}}^Z$ define a probability measure.

4.1 Model with foreign money market under the domestic risk-neutral measure

Consider the model for a risky asset, a foreign currency and a foreign money market introduced above in Section (2.1). Now add a domestic money market, with risk-free rate $R(t)$, $t \geq 0$.
Recall that the price at time t in *foreign currency* of one foreign money market unit is

$$M^f(t) = \exp\{\int_0^t R^f(u)\,du\}.$$

Given the exchange rate $Q(t)$, the price *in dollars* of a unit of the foreign money market is

$$N^f(t) = M^f(t)Q(t).$$

So there are 2 risky assets in this model:

$$\begin{aligned}
dS(t)\,dt &= \alpha(t)S(t)\,dt + \sigma_1(t)S(t)\,dW_1(t),\\
dN^f(t)\,dt &= \left[\gamma(t) + R^f(t)\right]N^f(t)\,dt\\
&\qquad + N^f(t)\sigma_2(t)\left[\rho(t)\,dW_1(t) + \sqrt{1-\rho^2(t)}\,dW_2(t)\right].
\end{aligned}$$

As definition (4.1) states, a domestic risk-neutral measure for this model must make $D(t)S(t)$ and $D(t)N^f(t) = D(t)M^f(t)Q(t)$ into martingales.

Remark:
Note that we *did not include the foreign exchange rate* $Q(t)$ into the set of risky assets to be considered. This is because $Q(t)$ can be thought of as the price of the foreign currency, not of the foreign money market. This is an important point:

given access to the foreign money market, $D(t)Q(t)$ should not be a martingale because holding foreign cash without depositing it at the foreign risk-free rate loses money; in any portfolio all foreign cash should be held in the foreign money market. Thus $N^f(t)$ is the appropriate asset to consider. Also note that its dynamics already incorporates the dynamics of $Q(t)$. This is in the same spirit as the discounted domestic currency should not be a martingale, but the discounted money market is, under the domestic risk neutral measure. Recall also in the classical Black-Scholes model, we considered 2 assets: the (domestic) money market and the stock S_t, not the (domestic) currency and the stock S_t.

Equation (17) in this case is

$$\begin{pmatrix} \sigma_1(t) & 0 \\ \sigma_2(t)\rho(t) & \sigma_2(t)\sqrt{1-\rho^2(t)} \end{pmatrix} \begin{pmatrix} \theta_1(t) \\ \theta_2(t) \end{pmatrix} = \begin{pmatrix} \alpha(t) - R(t) \\ \gamma(t) + R^f(t) - R(t) \end{pmatrix} \tag{20}$$

Assume there is a unique, risk-neutral measure for (9)–(10). By Theorem 2 equation (20) must then have a unique solution. Indeed it will, if $\sigma_1(t) > 0$, $\sigma_2(t) > 0$, and $-1 < \rho(t) < 1$ for all t with probability one. This solution is

$$\theta_1(t) = \frac{\alpha(t) - R(t)}{\sigma_1(t)}, \quad \theta_2(t) = \frac{1}{\sigma_2(t)\sqrt{1-\rho^2(t)}} \Big[\gamma(t) + R^f(t) - R(t) - \sigma_2(t)\rho(t)\theta_1(t)\Big].$$

Let $\widetilde{W}(t) = (W_1(t) + \int_0^t \theta_1(u)\,du, W_2(t) + \int_0^t \theta_2(u)\,du)$. Then one easily derives

$$\begin{aligned} dS(t) &= R(t)S(t)\,dt + \sigma_1(t)S(t)\,d\widetilde{W}_1(t) \\ dN^f(t) &= R(t)N^f(t)\,dt + N^f(t)\sigma_2(t)\left[\rho(t)\,d\widetilde{W}_1(t) + \sqrt{1-\rho^2(t)}\,d\widetilde{W}_2(t)\right]. \end{aligned}$$

5 Numéraires

Up to now, we have always assumed that prices were given in units of a fixed, domestic currency, which for concreteness we take to be US dollars. One could choose other units to measure prices, and it is often convenient, even necessary, to do so.

Let the price in dollars of some given asset or financial instrument be denoted $N(t)$. Let $S(t)$ be the price in dollars of any other asset. Then the ratio

$$S^{(N)} = \frac{S(t)}{N(t)}$$

is the price of the asset corresponding to S in units of the asset corresponding to N. In this situation, N is referred as the numéraire. The asset used for the numéraire could in principle be almost anything— a risky asset in the market, a foreign currency, a money market account, an index based on the market, or the price of a derivative.

Example 5.1. *Let $R(t)$ denote the (domestic) risk-free rate. It is common to think of $R(t)$ as the rate available from a money market account which can be added to or withdrawn from at will. One unit of a money market account is defined to be the value of 1\$ invested at time $t = 0$ and left in the account. This value in dollars at time t is $M(t) = \exp\{\int_0^t R(u)\,du\}$. If $S(t)$ is the price in dollars of an asset at time t, its price in units of the money market is*

$$\frac{S(t)}{\exp\{\int_0^t R(u)\,du\}} = e^{-\int_0^t R(u)\,du} S(t).$$

This is just the discounted price, or present value, of $S(t)$. So we can think of discounting as an example of pricing in money market account units.

Example 5.2. *(Non-example) Consider the market model studied in Examples 2 and 3. This consists of an asset with price (in dollars) $S(t)$, an exchange rate $Q(t)$ (dollars per unit of foreign currency), a domestic risk-free rate $R(t)$, and a foreign risk-free rate $R^f(t)$. Let $M(t) = \exp\{\int_0^t R(u)\,du\}$.*
There are many choices for denominating prices. A tempting example is to use the foreign currency as the numéraire. In this case, $S^{(Q)}(t) = S(t)/Q(t)$ is the price of the asset in units of the foreign currency, while a unit of the domestic money market in the foreign currency is $M^{(Q)}(t) = M(t)/Q(t)$. However, this should not be done, because under the domestic risk neutral measure, $Q(t)$ is NOT a martingale (See the discussion in section (4.1) and in section (5.1)). This is also consistent with our remark at the beginning of this note that we will only use non-dividend paying asset as numéraire. $Q(t)$, as denoting the price of the foreign currency, is a dividend paying asset with dividend rate R^f.

Example 5.3. *One could also use as numéraire the value in dollars $N^f(t) = M^f(t)Q(t)$ of a unit of the foreign money market. Then, in this unit*

$$S^{(N^f)}(t) = \frac{S(t)}{M^f(t)Q(t)} = e^{-\int_0^t R^f(u)\,du} S^{(Q)}(t).$$

is the price of the asset, and

$$\frac{Q(t)}{M^f(t)Q(t)} = e^{-\int_0^t R^f(u)\,du}$$

is the value of a unit of foreign currency.

5.1 Change of measure for change of numéraire

Risk-neutral pricing theory should not depend on the unit of price. If there is a risk-neutral measure when the price is in dollars, then there ought to be a risk-neutral measure $\widetilde{\mathbf{P}}^N$ for pricing with respect to N, for any numéraire N. This section addresses how to find $\widetilde{\mathbf{P}}^N$.
We will always start out with a risk-neutral model for $S(t) = (S_1(t), \ldots, S_d(t))$, given on a probability space $(\Omega, \mathcal{F}, \widetilde{\mathbf{P}})$, with filtration $\{\mathcal{F}(t);\, t \geq 0\}$, and a risk-free (domestic) rate $R(t)$, $t \geq 0$. As usual, $D(t) = \exp\{-\int_0^t R(u)\,, du\}$ denotes the discount factor. Thus we can also see this as starting out with *our default probability measure* as the domestic risk neutral measure.
Let $N(t)$, $t \geq 0$, be a strictly positive, numéraire process. Since $N(t)$ represents a price and the model is risk-neutral, $D(t)N(t)$, $t \geq 0$, is a martingale. In particular,

$$\tilde{E}\left[D(T)N(T)\right] = N(0), \quad \text{for any } T \geq 0. \tag{21}$$

Let T be the time horizon for which we want to study the market. It follows that

$$\widetilde{\mathbf{P}}^{(N)}(A) = \tilde{E}\left[\mathbf{1}_A \frac{D(T)N(T)}{N(0)}\right]$$

defines a new probability measure.

Theorem 3. $\widetilde{\mathbf{P}}^{(N)}$ *is a risk-neutral measure for pricing with respect to N in the following sense: for each i, $1 \leq i \leq m$, $S_i^{(N)}(t)$ is a martingale with respect to $\{\mathcal{F}(t);\, t \geq 0\}$ up to time T under $\widetilde{\mathbf{P}}^{(N)}$.*

Proof: The proof is an application of the formula for computing conditional expectations under a change of measure. Shreve states a special case in Lemma 5.2.2. The formula implies that for any sub-σ-algebra $\mathcal{G}$,

$$\tilde{E}^{(N)}\left[X \,\middle|\, \mathcal{G}\right] = \frac{\tilde{E}\left[X\frac{D(T)N(T)}{N(0)} \,\middle|\, \mathcal{G}\right]}{\tilde{E}\left[\frac{D(T)N(T)}{N(0)} \,\middle|\, \mathcal{G}\right]}. \tag{22}$$

In this formula $\tilde{E}^{(N)}$ represents expectation with respect to $\widetilde{\mathbf{P}}^{(N)}$. Apply equation (22) with $X = S_i^{(N)}(T)$ and $\mathcal{G} = \mathcal{F}(t)$. The result is

$$\tilde{E}^{(N)}\left[S_i^{(N)}(T) \,\middle|\, \mathcal{F}(t)\right] = \frac{\tilde{E}\left[\frac{S_i(T)}{N(T)}\frac{D(T)N(T)}{N(0)} \,\middle|\, \mathcal{F}(t)\right]}{\tilde{E}\left[\frac{D(T)N(T)}{N(0)} \,\middle|\, \mathcal{F}(t)\right]} = \frac{\tilde{E}\left[D(T)S_i(T) \,\middle|\, \mathcal{F}(t)\right]}{\tilde{E}\left[D(T)N(T) \,\middle|\, \mathcal{F}(t)\right]}.$$

But $D(u)S_i(u)$ and $D(u)N(u)$ are both martingales under $\widetilde{\mathbf{P}}$, and so
$\tilde{E}\left[D(T)S_i(T) \,\middle|\, \mathcal{F}(t)\right] = D(t)S_i(t)$ and $\tilde{E}\left[D(T)N(T) \,\middle|\, \mathcal{F}(t)\right] = D(t)N(t)$. Hence

$$\tilde{E}^{(N)}\left[S_i^{(N)}(T) \,\middle|\, \mathcal{F}(t)\right] = \frac{D(t)S_i(t)}{D(t)N(t)} = \frac{S_i(t)}{N(t)} = S_i^{(N)}(t).$$

This shows that $S_i^{(N)}(u)$, $0 \leq u \leq T$, is a martingale under $\widetilde{\mathbf{P}}^{(N)}$. ◇

5.2 Pricing under a change of numéraire

Suppose we have a financial product that pays $V(T)$ dollars at time T. Then the (domestic) risk neutral price of this product at time t is

$$V(t) = \tilde{E}\Big[\frac{D(T)}{D(t)}V(T)|\mathcal{F}(t)\Big]$$

(since $D(t)V(t)$ is a martingale under $\tilde{P}$).
What is the corresponding pricing formula when $V(t)$ is denoted under the unit of a numéraire $N(t)$? Arguing similar to the proof of Theorem (3) we will see that

$$\frac{V(t)}{N(t)} = \tilde{E}^{(N)}\Big[\frac{V(T)}{N(T)}|\mathcal{F}(t)\Big].$$

Thus denoting $V^{(N)}(t) := \frac{V(t)}{N(t)}$ we have

$$V^{(N)}(t) = \tilde{E}^{(N)}\Big[V^{(N)}(T)|\mathcal{F}(t)\Big].$$

This equation is meaningful by itself. It says that the price in the unit of numéraire $N(t)$ of the financial product is the conditional expectation under the corresponding risk neutral measure of the terminal value, also expressed in the same unit of numéraire. Note that the domestic risk neutral pricing formula is a special case of this when we use $N(t) = \frac{1}{D(t)}$, the domestic money market account.
It is also important to remember that here $V(t)$ is again in dollars, or the domestic currency, and $N(t)$ is the price in dollars of the numéraire of interest. To see a consequence of this, see the below section on pricing a financial product quoted in foreign currency.

5.3 Effect of change of numéraire

In section V, no assumptions were made on the nature of the price model. In this section, we specialize to the multi-asset model stated above and written under the risk-neutral measure as

$$d\,[D(t)S_i(t)] = D(t)S_i(t)\sum_{k=1}^{d}\sigma_{ik}(t)\,d\widetilde{W}_k(t), \qquad 1 \le i \le m. \tag{23}$$

In addition, we impose the assumption that $\{\mathcal{F}(t);\, t \ge 0\}$ is generated by $\widetilde{W}$.
Let N be a num'eraire process and $\widetilde{\mathbf{P}}^{(N)}$ the risk-neutral measure for N. $\widetilde{W}$ is no longer a Brownian motion under $\widetilde{\mathbf{P}}^{(N)}$. The object of this section is use Theorem 1 and Girsanov's theorem to identify an appropriate Brownian motion $\widetilde{W}^{(N)}(t)$ under $\widetilde{\mathbf{P}}^{(N)}$ and to rewrite equation (23) using it.
Now $D(t)N(t)$ is a martingale under $\widetilde{\mathbf{P}}$, and $N(t)$, as a numéraire, is strictly positive for all t. So Theorem 1 implies there is a process ν so that

$$\frac{D(t)N(t)}{N(0)} = \exp\{\int_0^t \nu(u)\cdot d\widetilde{W}(u) - \frac{1}{2}\int_0^t \|\nu(u)\|^2\,du\} \tag{24}$$

Since $\widetilde{\mathbf{P}}^{(N)}(A) = \tilde{E}[\mathbf{1}_A\frac{D(T)N(T)}{N(0)}]$, it also follows from Theorem 1 that $\widetilde{W}^{(n)}(t) = \widetilde{W}(t) - \int_0^t \nu(u)\,du$ is a Brownian motion under $\widetilde{\mathbf{P}}^{(N)}$ up to time T.
Define $\sigma_i(t) = (\sigma_{i1}(t),\ldots,\sigma_{id}(t))$, so that (23) may be written compactly as

$$dD(t)S_i(t) = D(t)S_i(t)\left[\sigma_i(t)\cdot d\widetilde{W}(t)\right].$$

By (11), the solution to this equation is

$$D(t)S_i(t) = S_i(0)\exp\{\int_0^t \sigma_i(u)\cdot d\widetilde{W}(u) - \frac{1}{2}\int_0^t \|\sigma_i(u)\|^2\,du\}.$$

Thus, using the representation (24) for $D(t)N(t)$,

$$\begin{aligned} S_i^{(N)}(t) &= \frac{S_i(t)}{N(t)} = \frac{D(t)S_i(t)}{D(t)N(t)} \\ &= \frac{S_i(0)}{N(0)}\exp\{\int_0^t [\sigma_i(u)-\nu(u)]\cdot d\widetilde{W}(u) - \frac{1}{2}\int_0^t (\|\sigma_i(u)\|^2 - \|\nu(u)\|^2)\,du\} \end{aligned}$$

Replace $d\widetilde{W}$ in this expression by $d\widetilde{W}^{(N)}(t) + \nu(t)\,dt$. Note first that

$$\begin{aligned} &\exp\{\int_0^t [\sigma_i(u)-\nu(u)]\cdot[d\widetilde{W}^{(N)}(u) + \nu(u)\,du] \\ &= \int_0^t [\sigma_i(u)-\nu(u)]\cdot d\widetilde{W}^{(N)}(u) + \int_0^t \sigma_i(u)\cdot\nu(u)\,du - \int_0^t \nu(u)\cdot\nu(u)\,du \\ &= \int_0^t [\sigma_i(u)-\nu(u)]\cdot d\widetilde{W}^{(N)}(u) + \int_0^t \sigma_i(u)\cdot\nu(u)\,du - \int_0^t \|\nu(u)\|^2\,du \end{aligned}$$

It follows that

$$S_i^{(N)}(t) = \frac{S_i(0)}{N(0)} \exp\{\int_0^t [\sigma_i(u) - \nu(u)] \cdot d\widetilde{W}^{(N)}(u) - \frac{1}{2}\int_0^t (\|\sigma_i(u)\|^2 - 2\underline{\sigma}_i(u)\cdot\nu(u) + \|\nu(u)\|^2)\, du\}.$$

But

$$\begin{aligned} \|\underline{\sigma}_i(u) - \nu(u)\|^2 &= [\sigma_i(u) - \nu(u)] \cdot [\sigma_i(u) - \nu(u)] \\ &= \|\sigma_i(u)\|^2 - 2\sigma_i(u)\cdot\nu(u) + \|\nu(u)\|^2) \end{aligned}$$

Thus

$$S_i^{(N)}(t) = \frac{S_i(0)}{N(0)} \exp\{\int_0^t [\sigma_i(u) - \nu(u)] \cdot d\widetilde{W}^{(N)}(u) - \frac{1}{2}\int_0^t \|\sigma_i(u) - \nu(u)\|^2\, du\}$$

It follows from equation (11) that

$$dS_i^{(N)}(t) = S_i^{(N)}(t)[\sigma_i(t) - \nu(t)] \cdot d\widetilde{W}^{(N)}(t) = S_i^{(N)}(t) \sum_{k=1}^{d} (\sigma_{ik}(t) - \nu_k(t))\, d\widetilde{W}_k^{(N)}(t) \quad (25)$$

This is an interesting equation because it shows exactly how the volatility of $S_i^{(N)}$ differs from that of S_i. Of course, we expect them to differ because N itself has volatility and $S_i^{(N)}(t) = S_i(t)/N(t)$. In fact, from the expression (24) and from (11) one finds that

$$dN(t) = R(t)N(t)\, dt + \sum_{k=1}^{d} \nu_k(t)\, d\widetilde{W}_k(t),$$

so $\nu_k(t)$ is the component of the volatility of N at time t due to $\widetilde{W}_k$.

6 Foreign risk-neutral measure

The discussion of the previous section established the existence of ν, but not a formula for it. In examples it can be found explicitly if the numéraire is defined explicitly.

Consider the example of a market with an asset and foreign currency formulated above. Its risk neutral version was derived in Example 3 and is

$$\begin{aligned} dS(t) &= R(t)S(t)\, dt + \sigma_1(t)S(t)\, d\widetilde{W}_1(t) && (26) \\ dN^f(t) &= R(t)N^f(t)\, dt + N^f(t)\sigma_2(t)\left[\rho(t)\, d\widetilde{W}_1(t) + \sqrt{1-\rho^2(t)}\, d\widetilde{W}_2(t)\right]. && (27) \end{aligned}$$

Recall that $N^f(t) = \exp\{\int_0^t R^f(u)\,du\}Q(t)$ is the dollar value of one unit of **the foreign money market account**. *We shall use it as the numéraire in this section.* The domestic discount factor is $D(t) = \exp\{-\int_0^t R(u)\,du\}$.
From (27),

$$\begin{aligned} d[D(t)N^f(t)] &= D(t)N^f(t)\sigma_2(t)\left[\rho(t)\,d\widetilde{W}_1(t) + \sqrt{1-\rho^2(t)}\,d\widetilde{W}_2(t)\right] \\ &= D(t)N^f(t)\left(\sigma_2(t)\rho(t), \sigma_2(t)\sqrt{1-\rho^2(t)}\right)\cdot d\widetilde{W}(t). \end{aligned}$$

Note: $D(0)N^f(0) = Q(0)$. It follows from (11) that

$$D(t)N^f(t) = Q(0)\exp\{\int_0^t \nu(u)\cdot d\widetilde{W}(t) - \frac{1}{2}\int_0^t \|\nu(u)\|^2\,du\},$$

where $\nu(t) = \left(\sigma_2(t)\rho(t), \sigma_2(t)\sqrt{1-\rho^2(t)}\right)$.
The risk-neutral measure for denominating prices in units of the foreign money market up to time T, or *the foreign risk neutral measure* in short, is therefore defined by

$$\widetilde{\mathbf{P}}^{(N^f)}(A) = \tilde{E}\left[\mathbf{1}_A\frac{D(T)N^f(T)}{Q(0)}\right] = \tilde{E}\left[\mathbf{1}_A\exp\{\int_0^t \nu(u)\cdot d\widetilde{W}(t) - \frac{1}{2}\int_0^t \|\nu(u)\|^2\,du\}\right],$$

and

$$\widetilde{W}^{(N^f)}(t) = \left(\widetilde{W}_1(t) - \int_0^t \sigma_2(u)\rho(u)\,du,\ \widetilde{W}_2(t) - \int_0^t \sigma_2(u)\sqrt{1-\rho^2(u)}\,du\right)$$

is a Brownian motion up to time T under $\widetilde{\mathbf{P}}^{(N^f)}$.
The price of the asset with respect to numéraire $N^f(t)$ is $S^{(N^f)}(t) = S(t)/N^f(t)$. It is the *present value* of the asset price in units of the foreign currency (or just simply the value of the asset price at time t in units of the foreign money market). It is a martingale with respect to $\widetilde{\mathbf{P}}^{(N^f)}$. By applying (25),

$$dS^{(N^f)}(t) = S^{(N^f)}(t)\left[(\sigma_1(t) - \sigma_2(t)\rho(t))\,d\widetilde{W}_1^{(N^f)}(t) - \sigma_2(t)\sqrt{1-\rho^2(t)}\,d\widetilde{W}_2^{(N^f)}(t)\right], \quad t \le T.$$

Because $\widetilde{W}_2$ contributes to the volatility of $N^f(t)$, $d\widetilde{W}_2^{(N^f)}(t)$ contributes to the volatility of $S^{(N^f)}(t)$.

6.1 Pricing a financial product quoted in foreign currency

Suppose we have a financial product that pays $V(T)$ *units of foreign currency* at time T. Then we have the following lemma

Lemma 6.1. *The risk neutral price (in foreign currency) of the above product is*

$$V(t) = \tilde{E}^{(N^f)}\Big(e^{-\int_t^T R^f(u)du}V(T)\Big|\mathcal{F}(t)\Big).$$

Note that the result is very intuitive: to price a financial product quoted in foreign currency, we take conditional expectation under the foreign risk neutral measure, discounted under the foreign interest rate.

Proof. The proof of this Lemma relies on the result of Section (5.2). First, we need to convert $V(T)$ into dollars. Thus the financial product pays $V(T)Q(T)$ dollars at time T. Now, since the product is quoted in foreign currency, *we need to use the foreign money market numéraire* $N^f(t)$. Thus we have

$$V^{(N^f)(t)} = \tilde{E}^{(N^f)}\Big[\frac{V(T)Q(T)}{N^f(T)}\Big|\mathcal{F}(t)\Big].$$

Note that $N^f(T) = M^f(T)Q(T)$, and $V^{(N^f)(t)} = \frac{V(t)}{M^f(t)}$ (be careful to distinguish $V(t)$ here and the $V(t)$ in Section (5.2). The $V(t)$ here is the risk neutral price in foreign currency. The $V(t)$ in Section (5.2) is the risk neutral price in domestic currency. They are unrelated).

After simplifying, we get

$$\frac{V(t)}{M^f(t)} = \tilde{E}^{(N^f)}\Big[\frac{V(T)}{M^f(T)}\Big|\mathcal{F}(t)\Big].$$

Since $M^f(t) = e^{\int_0^t R^f(u)du}$ the conclusion follows.

Alternatively, the risk neutral price (in dollars) of this financial product is

$$\tilde{V}_t = \tilde{E}\Big[\frac{D(T)V(T)Q(T)}{D(t)}\Big|\mathcal{F}(t)\Big].$$

But we have

$$\begin{aligned}\tilde{E}\Big[\frac{D(T)V(T)Q(T)}{D(t)}\Big|\mathcal{F}(t)\Big] &= \tilde{E}^{(N^f)}\Big[\frac{D(T)V(T)Q(T)}{D(t)}\frac{Q(t)M^f(t)D(t)}{Q(T)M^f(T)D(T)}\Big|\mathcal{F}(t)\Big] \\ &= \tilde{E}^{(N^f)}\Big[\frac{D^f(T)V(T)Q(t)}{D^f(t)}\Big|\mathcal{F}(t)\Big].\end{aligned}$$

Thus dividing by $Q(t)$ on both sides gives

$$D^f(t)V(t) = \tilde{E}^{(N^f)}\Big[D^f(T)V(T)\Big|\mathcal{F}(t)\Big].$$